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Performance Analysis of an Optimum Detector with Counting Point Process Observations

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ABSTRACT

Transformations of the classical hypothesis testing problem with continuous-time observations to a hypothesis testing problem with counting point process observations are considered. A general form for the random intensity rate (RIR) which can accommodate feedback is investigated and the optimal solution in the Neyman-Pearson sense is specified. An analysis of the performance of the corresponding optimum processor is studied.

I. Introduction

In [1] and [2] we considered the classical hypothesis testing problem

$$\begin{aligned} H_0: & \text{under } \bar{P}_0, (X_t), 0 \leq t \leq T, \text{ is a Wiener process,} \\ H_1: & \text{under } \bar{P}_1, (X_t - \int_0^t S_u du), 0 \leq t \leq T, \text{ is a Wiener process,} \end{aligned} \quad (1)$$

where $X = (X_t, F_t \otimes \bar{F}_0)$, $0 \leq t \leq T$, are the continuous-time observations, and investigated transformations of this test into the hypothesis testing problem

$$\begin{aligned} H_0: & \text{under } \bar{P}_0, (N_t), 0 \leq t \leq T, \text{ is a DSPP with RIR } (\hat{\lambda}_t^{(0)}), 0 \leq t \leq T, \\ H_1: & \text{under } \bar{P}_1, (N_t), 0 \leq t \leq T, \text{ is a DSPP with RIR } (\hat{\lambda}_t^{(1)}), 0 \leq t \leq T, \end{aligned} \quad (2)$$

where $N = (N_t, F_t \otimes \bar{F}_t)$, $0 \leq t \leq T$, are the counting point process observations. In addition, several suboptimum detection schemes were presented.

The aim of the investigations in [1] and [2] was to find, from among a given class of codes, the RIR λ as a functional of the "information bearing process" X and the observations N , i.e.,

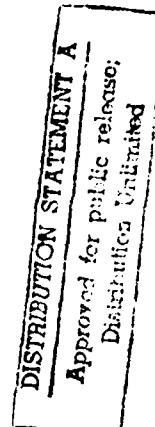
$$\lambda_t = \lambda_t(X, N), \quad (3)$$

such that we attain or come close to some "optimum properties".

In this paper we extend the results of [1] and [2] by finding, from the class of σ -admissible codes, the Neyman-Pearson optimum RIR λ . The paper is organized as follows. In section II we consider the classical "input" hypothesis testing problem of deciding between two absolutely continuous measures \bar{P}_0 and \bar{P}_1 ($\bar{P}_1 \ll \bar{P}_0$), given some observation process $X = (X_t, F_t \otimes \bar{F}_0)$, $0 \leq t \leq T$. In the sequel the corresponding Radon-Nikodym derivative, denoted by Λ_T , will be called the input likelihood ratio. We also consider the "output" hypothesis testing problem of deciding between the probability measures \bar{P}_0 and \bar{P}_1 given the counting point

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process observations $N = (N_i, F_0 \otimes \bar{F}_i)$, $0 \leq i \leq T$. The corresponding Radon-Nikodym derivative will be called the output likelihood ratio. Theorem 1 specifies the Neyman-Pearson optimum RIR (of significance level α_0) for detection in the class of o-admissible codes. It is shown that, if the causality condition on λ is dropped, the optimal RIR is given by

$$\lambda_i = \mu_0 + b \cdot \chi_{[0,L]}(\Lambda_T), \quad (4)$$

for all i , $0 \leq i \leq T$, where b is constant and $\chi_{[0,L]}$ is the characteristic function of the set $[0, L]$. We conjecture that if λ is nonanticipative the RIR is given by (for more details see section II)

$$\lambda_i = \mu_0 + b \cdot \chi_{[0,L]}(\Lambda_i), \quad (5)$$

for all i , $0 \leq i \leq T$.

The degradation which appears in the output detection problem by using the optimum RIR λ (introduced by the counting point process observations) is established in section III. The main result in section III is presented in Theorem 2. It relates (see also Lemma 3) the power and the probability of false alarm of a very general hypothesis testing problem when its sample space is not directly observable (DSPP observations) to the same parameters α and β one would obtain if the input space were directly observable.

II. The Optimal Random Intensity Rate for Detection

Let us consider the "input" hypothesis testing problem of deciding between two absolutely continuous probability measures \bar{P}_0 and \bar{P}_1 ($\bar{P}_1 \ll \bar{P}_0$), defined on the measurable space $(\bar{\Omega}, \bar{\mathcal{F}})$, given the observation process $X = (X_i, F_i \otimes \bar{F}_0)$, $0 \leq i \leq T$. The construction of the probability measures \bar{P}_0 and \bar{P}_1 is given in [3]. The corresponding Neyman-Pearson test of significance level α_0 is completely characterized by the sufficient statistic $\chi_{[0,L]}(\Lambda_T)$, where L is the associated threshold [4].

Let us consider the "output" hypothesis testing problem of deciding between the probability measures \bar{P}_0 and \bar{P}_1 given the observation counting point process $N = (N_i, F_0 \otimes \bar{F}_i)$, $0 \leq i \leq T$. The corresponding Neyman-Pearson test of significance level α_0 is completely characterized by $\tilde{\chi}_{[0,L]}(\Lambda_T)$ where L is the threshold .

Definition 1. The class of codes satisfying the peak and average power constraint

$$0 \leq \lambda_i(X, N) \leq c \quad \text{and} \quad \bar{E}_i \frac{1}{T} \int_0^T \lambda_s ds \leq qc, \quad (6)$$

for all i and i , $0 \leq i \leq T$ and $i=0,1$ and constants c and q , $c \in \mathbb{R}_+$ and $0 < q \leq 1$ is called o-admissible.

Definition 2. The coding $\lambda = (\lambda_i(X, N), F_i \otimes \bar{F}_i)$, $0 \leq i \leq T$, is said to be optimum in the Neyman-Pearson sense of significance level α_0 if over the class of o-admissible codes the supremum

$$\sup_{0 \leq \lambda_i \leq c, \bar{E}_i \frac{1}{T} \int_0^T \lambda_s ds \leq qc} \hat{\beta} \quad (7)$$

is achieved for all α , $\alpha \leq \alpha_0$, i.e., the coding λ maximizes the power among all Neyman-Pearson tests.

Remark. The optimal RIR λ in the Neyman-Pearson sense features the best performance for only a given probability of false alarm α_0 . Therefore the above optimality criterion leads in general to solutions which are "locally" optimal. In other words for each α_0 a different code λ might be obtained.

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Due to analytical difficulties we will restrict the following analysis to the case where λ , is measurable with respect to the σ -algebra $F_T \otimes \bar{F}_t$, for all t , $0 \leq t \leq T$.

Theorem 1. The RIR $\lambda = (\lambda_t, F_t \otimes \bar{F}_t)$, $0 \leq t \leq T$, corresponding to the (output observation) counting point process \bar{N} is optimum in the Neyman-Pearson sense if it is given by

$$\lambda_t(X, N) = \mu_0 + b \cdot \chi_{[0, L]}(\Lambda_T), \quad (8)$$

where $b = (1 \wedge \frac{\beta}{\alpha})c$, for all t , $0 \leq t \leq T$.

A proof of this theorem appears in [3].

Remark. The above theorem implies that, under the Neyman-Pearson optimality criterion, feedback does not increase the power of the output hypothesis testing problem. This result is somewhat similar to the result obtained on the capacity of the Poisson type channel under a peak and average power constraint (see also [5]-[7]).

Since

$$\hat{\lambda}_s^{(0)} = \frac{1}{1 + \frac{\alpha}{1-\alpha} e^{bs} \left(\frac{\mu_0}{\mu_0 + b} \right)^{N_s}}, \quad (9)$$

and

$$\hat{\lambda}_s^{(1)} = \frac{1}{1 + \frac{\beta}{1-\beta} e^{bs} \left(\frac{\mu_0}{\mu_0 + b} \right)^{N_s}}, \quad (10)$$

for all s , $0 \leq s \leq T$, we conclude that the test statistic for the output hypothesis testing problem is given by

$$\tilde{\Lambda}_T \begin{cases} \geq \tilde{L} & \rightarrow H_1 \\ < \tilde{L} & \rightarrow H_0, \end{cases} \quad (11)$$

where

$$\begin{aligned} \tilde{\Lambda}_T = & \int_0^T \ln \frac{\frac{\mu_0 + b}{1 + \frac{\beta}{1-\beta} e^{bs} \left(\frac{\mu_0}{\mu_0 + b} \right)^{N_s}}}{\frac{\mu_0 + b}{1 + \frac{\alpha}{1-\alpha} e^{bs} \left(\frac{\mu_0}{\mu_0 + b} \right)^{N_s}}} dN_s \\ & - b \int_0^T \left[\frac{1}{1 + \frac{\beta}{1-\beta} e^{bs} \left(\frac{\mu_0}{\mu_0 + b} \right)^{N_s}} - \frac{1}{1 + \frac{\alpha}{1-\alpha} e^{bs} \left(\frac{\mu_0}{\mu_0 + b} \right)^{N_s}} \right] ds. \end{aligned} \quad (12)$$

For the case where λ , is measurable with respect to the σ -algebra $F_t \otimes \bar{F}_t$, for all t , $0 \leq t \leq T$, we state the following

Conjecture 1. The RIR $\lambda = (\lambda_t, F_t \otimes \tilde{F}_t)$, $0 \leq t \leq T$, corresponding to the (output observation) counting point process N is optimum in the Neyman-Pearson sense if it is given by

$$\lambda_t(X, N) = \mu_0 + b \cdot \chi_{[0, L_t]}(\Lambda_t), \quad (13)$$

where $b = (1 \wedge \frac{q}{\beta \vee \alpha_0})c$, for all t , $0 \leq t \leq T$. The threshold $L : R_+ \rightarrow R_+$ is defined by[†]

$$\bar{P}_0(\Lambda_t \geq L_t) = \alpha_0.$$

Remark. The RIR λ described by (13) can be seen as the causal version of the RIR λ given by (8).

In the next section an analysis of the performance of the optimum detection scheme is presented. It is shown that an equivalent form of (12) can be given that allows a simple analysis of the performance of the above optimum detection scheme.

III. Performance Analysis of the Optimal Detection Scheme

The performance analysis of the optimum Neyman-Pearson detector (12) is greatly simplified by the following:

Lemma 1. If $\beta \geq \alpha$ and $\lambda_s = \mu_0 + b \cdot \chi(\Lambda_T)$ for every s , $0 \leq s \leq T$, then[†]

$$\tilde{\Lambda}_T = \tilde{\Lambda}_T(N_T), \quad (14)$$

where $\tilde{\Lambda}_T$ is a nonincreasing Borel function of N_T .

A proof of this lemma based on the abstract Bayes formula [8] is given in [3].

Since $\tilde{\Lambda}_T = \tilde{\Lambda}_T(N_T)$ is a nonincreasing function, the Neyman-Pearson test

$$\tilde{\Lambda}_T \begin{cases} \geq \tilde{L} & \rightarrow H_1 \\ < \tilde{L} & \rightarrow H_0. \end{cases} \quad (15)$$

(where \tilde{L} is the threshold of the output test) is equivalent to

$$N_T \begin{cases} < l & \rightarrow H_1 \\ \geq l & \rightarrow H_0 \end{cases} \quad (16)$$

(where l is the corresponding new threshold of the output test).

Since $\tilde{\Lambda}_T$ is a nonincreasing function of the observations, the power $\tilde{\beta}$ and the level $\tilde{\alpha}$ of the output hypothesis testing problem with counting point process observations can be reduced to

$$\begin{aligned} \tilde{\beta} &= \bar{P}_1(N_T < l) = \sum_{k=0}^{l-1} \bar{P}_1(N_T = k), \\ \tilde{\alpha} &= \bar{P}_0(N_T < l) = \sum_{k=0}^{l-1} \bar{P}_0(N_T = k), \end{aligned} \quad (17)$$

[†] Randomized tests are not considered here. The extension to randomized tests can be done in the usual manner [4].

where $\bar{\beta}$ is the new threshold.

In the following, with straightforward mathematics, we will find an equivalent expressions for $\bar{\beta}$ and $\bar{\alpha}$. Our standing assumption is that the observation process $N = (N_t, F_0 \otimes \bar{F}_t)$, $0 \leq t \leq T$, is a doubly stochastic Poisson process (DSPP).

Let us first relate the power β and the probability of false alarm α of the input to the power $\bar{\beta}$ and probability of false alarm $\bar{\alpha}$ of the output. We can give the following

Lemma 2.

$$\begin{aligned}\bar{\beta} &= \sum_{k=0}^{l-1} \frac{[(\mu_0+b)]^k}{k!} \exp[-(\mu_0+b)T] - \sum_{k=0}^{l-1} \frac{T^k}{k!} [(\mu_0+b)^k \exp(-bT) - \mu_0^k] \exp(-\mu_0 T) \beta, \\ \bar{\alpha} &= \sum_{k=0}^{l-1} \frac{[(\mu_0+b)]^k}{k!} \exp[-(\mu_0+b)T] - \sum_{k=0}^{l-1} \frac{T^k}{k!} [(\mu_0+b)^k \exp(-bT) - \mu_0^k] \exp(-\mu_0 T) \alpha.\end{aligned}\quad (18)$$

Proof. Both $\bar{\beta}$ and $\bar{\alpha}$ can be obtained with some easy manipulations, and only those for $\bar{\beta}$ are presented.

$$\begin{aligned}\bar{\beta} &= \sum_{k=0}^{l-1} \bar{P}_1(N_T=k) = \sum_{k=0}^{l-1} \frac{T^k}{k!} \bar{E}_1[(\mu_0+b \cdot \chi_{[0,L]}(Z))^k \exp[-(\mu_0+b \cdot \chi_{[0,L]}(Z))T]] \\ &= \sum_{k=0}^{l-1} \frac{T^k}{k!} \{\mu_0^k \exp(-\mu_0 T) + [(\mu_0+b)^k \exp[-(\mu_0+b)T] - \mu_0^k \exp(-\mu_0 T)](1-\beta)\} \\ &= \sum_{k=0}^{l-1} \frac{[(\mu_0+b)]^k}{k!} \exp[-(\mu_0+b)T] - \sum_{k=0}^{l-1} \frac{T^k}{k!} [(\mu_0+b)^k \exp(-bT) - \mu_0^k] \exp(-\mu_0 T) \beta.\end{aligned}\quad \square$$

Let us compare the pairs (β, α) and $(\bar{\beta}, \bar{\alpha})$ in the limiting case $\mu_0=0$. The test statistic can be easily reduced to

$$N_T \left\{ \begin{array}{ll} = 0 & \rightarrow H_1 \\ \geq 1 & \rightarrow H_0. \end{array} \right. \quad (19)$$

The power $\bar{\beta}$ and the probability of false alarm $\bar{\alpha}$ can also be computed directly:

$$\begin{aligned}\bar{\beta} &= \bar{P}_1(N_T=0) = e^{-bT} + (1-e^{-bT})\beta, \\ \bar{\alpha} &= \bar{P}_0(N_T=0) = e^{-bT} + (1-e^{-bT})\alpha.\end{aligned}\quad (20)$$

Fig. 1 illustrates graphically the results that we might expect in terms of the receiver operating characteristic (ROC). Since $\beta=\bar{\beta}$ implies $\beta=1$, the only point of intersection of the above curves occurs for $(\alpha, \beta)=(1, 1)$.

Another useful representation for $\bar{\alpha}$ and $\bar{\beta}$ is given in

Lemma 3.

$$\begin{aligned}\beta &= 1 - \bar{E}_1 \chi_{[0,L]}(Z), \\ \bar{\beta} &= 1 - \bar{E}_1 \left[\sum_{k \geq l} \frac{(\mu_0 T)^k}{k!} \exp(-\mu_0 T) - \sum_{k=0}^{l-1} \frac{T^k}{k!} [(\mu_0+b)^k \exp(-bT) - \mu_0^k] \exp(-\mu_0 T) \chi_{[0,L]}(Z) \right].\end{aligned}\quad (21)$$

¹ By abuse of notation.

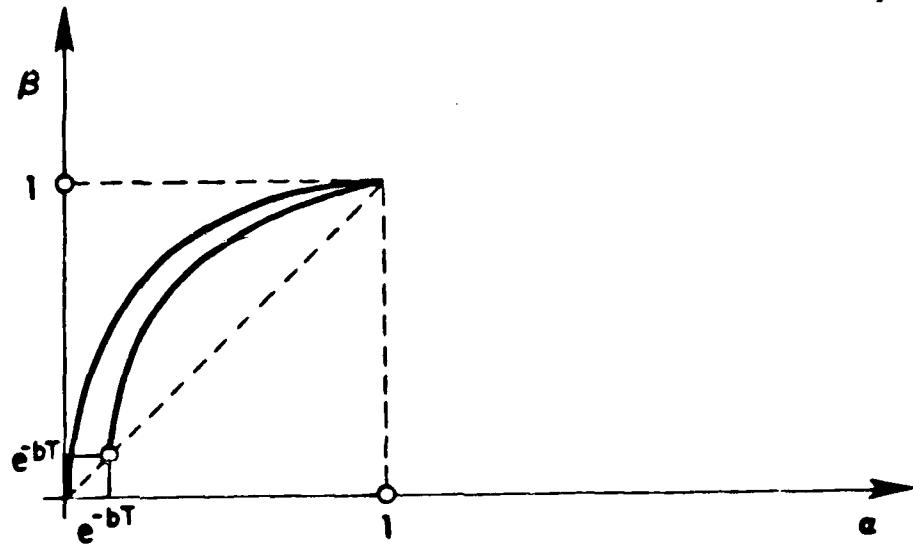


Fig. 1 The receiver operating characteristic (optimum processor).

$$\alpha = 1 - \bar{E}_0 \chi_{[0,L]}(Z),$$

$$\tilde{\alpha} = 1 - \bar{E}_0 \left[\sum_{k \geq 1} \frac{(\mu_0 T)^k}{k!} \exp(-\mu_0 T) - \sum_{k=0}^{l-1} \frac{T^k}{k!} [(\mu_0 + b)^k \exp(-bT) - \mu_0^k \exp(-\mu_0 T)] \chi_{[0,L]}(Z) \right], \quad (22)$$

where $\chi_{[0,L]}(Z)$ denotes the characteristic function of the set $[0,L]$ on the real line.

Proof. The proof can be easily supplied. \square

If we define the function $\kappa_{opt}: R_+ \rightarrow [0,1]$ by

$$\begin{aligned} \kappa_{opt}(\zeta) &= \sum_{k \geq 1} \frac{T^k}{k!} \mu_0^k \exp(-\mu_0 T) \\ &\quad - \sum_{k=0}^{l-1} \frac{T^k}{k!} [(\mu_0 + b)^k \exp[-(\mu_0 + b)T] - \mu_0^k \exp(-\mu_0 T)] \chi_{[0,L]}(\zeta), \end{aligned} \quad (23)$$

for all $\zeta, \zeta \in R_+$, we arrive at the following fundamental result:

Theorem 2. The optimum detector at the output of the processor[†] performs a randomized test given by

$$\kappa_{opt}(\zeta), \quad (24)$$

which is an approximation to the optimum input test specified by

$$\chi_{[0,L]}(\zeta), \quad (25)$$

for all $\zeta, \zeta \in R_+$, respectively. The limiting case where $\mu_0=0$ can easily be analyzed. We have

$$\begin{aligned} \hat{\beta} &= 1 - \bar{E}_1 \chi_{[0,L]}(Z)(1 - e^{-bT}), \\ \tilde{\alpha} &= 1 - \bar{E}_0 \chi_{[0,L]}(Z)(1 - e^{-bT}), \end{aligned} \quad (26)$$

[†] See [1]-[3] for the definition of the processor.

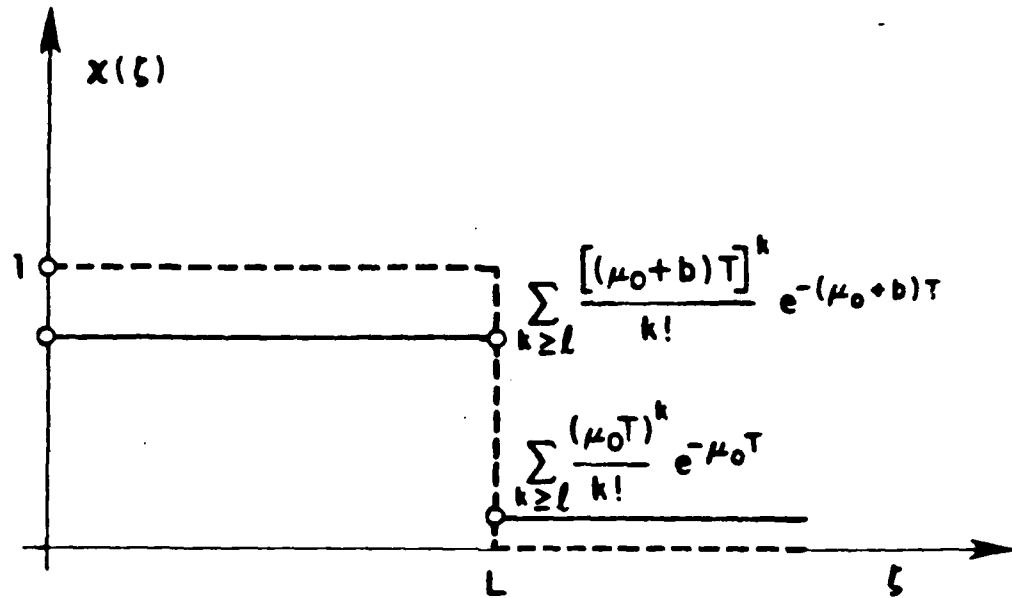


Fig. 2 Representation of the optimum input and optimum output test.

and thus the test statistic is given by

$$\kappa_{opt}(\zeta) = (1 - e^{-bT}) \chi_{[0, L]}(\zeta), \quad (27)$$

for all $\zeta, \zeta \in R_+$. The results obtained are depicted in Fig. 2.

The decision procedure at the output of the processor is strictly suboptimal when compared to the optimal decision procedure at the input, should the input be completely observable. This is true since, under the Neyman-Pearson criterion, any randomized test will not perform as well as the corresponding optimum test given by (25) [9, p.65]. This agrees with our intuitive feeling that the transformation which takes place through the processor (see also Fig. 2.1 in [3]) affects the information.

Remark. The randomized test κ_{opt} has a very simple probabilistic interpretation. According to this interpretation, for a particular value of the input likelihood ratio ζ , with probability $\kappa_{opt}(\zeta)$ we decide that the null-hypothesis is true and with probability $1 - \kappa_{opt}(\zeta)$ we decide that the alternative is true. The deviation between the optimum randomized test $\chi_{[0, L]}$ and the randomized test κ_{opt} will define a measure for evaluating the performance of the processor.

IV. Conclusion

In this paper the Neyman-Pearson optimum RIR λ from among the class of α -admissible codes has been specified. Since the problem of finding the ROC even for simple hypothesis testing problems is known to be analytically intractable, a new type of comparison between the optimum and a suboptimum processor has been introduced. In particular, the output Neyman-Pearson test compared to the similar test for the input turns out to be a randomized test. The latter test shows us to what degree the processor specified by the RIR λ leads to a loss of "information". This loss supports our intuitive feeling that the "stochastic mapping" describing the processor has to reduce the amount of "information" available at its input.

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